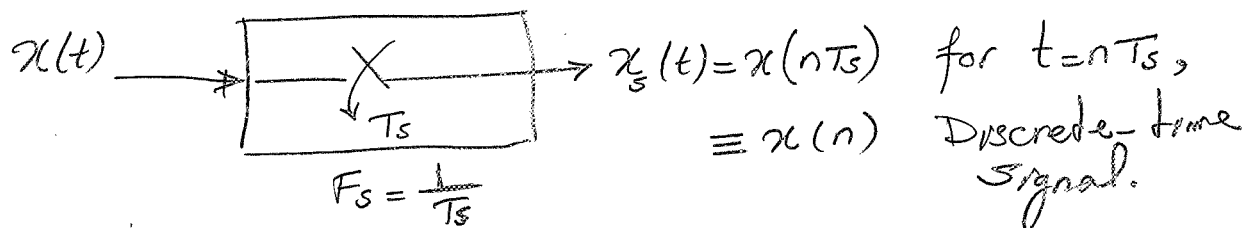
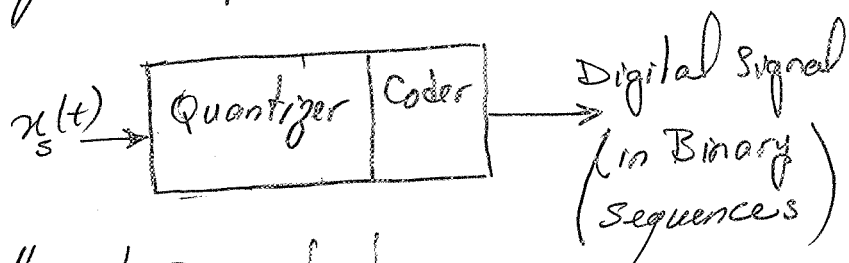


Discrete-time Signals and Systems



"Ideal, Uniform Sampler"



* We deal here with only Discrete-time or sampled signals and systems.

Classification of Discrete-time Signals

* Energy Signals $x(n)$ is Energy signal

if

$$\text{Energy of } x(n) = E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

If $x(n)$ is real, $x(n) = x^*(n)$, then

$$E_x = \sum_{n=-\infty}^{\infty} x^2(n).$$

Partial Energy $E_x^N = \sum_{n=-N}^N |x(n)|^2$

Thus $E_x = \lim_{N \rightarrow \infty} E_x^N = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x(n)|^2$

* Power Signals

(1) If $x(n)$ is not Energy signal, i.e. $E_x = \infty$, then $x(n)$ may or may not be a power signal

(2) $x(n)$, with infinite energy, is power signal

if

$$P_x = \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \right) E_x^N = \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \right) \sum_{n=-N}^N |x(n)|^2$$

exists, i.e. is finite and $< \infty$.

* Periodic signals

A periodic signal is a power signal such that it is periodic, that is

$$x(n) = x(n+N) = x(n+kN)$$

where N is smallest integer for which $x(n) = x(n+N)$.

N is called Fundamental period.

* If there is no integer N , then $x(n)$ is "Aperiodic" signal.

* $P_{x \text{ periodic}} = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$ for $x(n) = x(n+N)$

Ex1 Let $x(t) = A \cos(2\pi F_0 t + \theta)$, where F_0 (in Hz) is the known frequency of $x(t)$.

(1) We sample $x(t)$ at sampling rate of F_s (Hz).
Find the sampled signal (i.e. Discrete-time signal)

$x(n)$.

(2) Determine if $x(n)$ is periodic, and find N if $x(n) = x(n+N)$.

Sol

part (1): $x(t) \Big|_{t=nT_s} = A \cos(2\pi F_0 nT_s + \theta)$

$$F_s = \frac{1}{T_s}$$

$$x(n) = x(nT_s) = A \cos\left(2\pi \frac{F_0}{F_s} n + \theta\right)$$

We define
cycle (or Hz)
per sample

$$f_0 = \frac{F_0}{F_s} \xrightarrow{\text{Hz}}, \quad 2\pi f_0 = \frac{2\pi F_0}{F_s} = \Omega_0$$

$$\Omega_0 = 2\pi F_0 T_s, \quad \omega_0 = 2\pi F_0, \quad \Omega_0 = \omega_0 T_s$$

radian per sample
Second
radian per second

Thus

$$x(nT_s) = x(n) = A \cos(2\pi f_0 n + \theta) = A \cos(\Omega_0 n + \theta)$$

Part (2): we need to check if $x(n) = x(n+N)$.

$$x(n) = A \cos(2\pi f_0 n + \theta)$$

$$x(n+N) = A \cos(2\pi f_0 n + \theta + 2\pi f_0 N)$$

$$= A \left\{ \cos(2\pi f_0 n + \theta) \cos(2\pi f_0 N) - \sin(2\pi f_0 n + \theta) \sin(2\pi f_0 N) \right\}$$

If $2\pi f_0 N = 2\pi k$, $k = 0, \pm 1, \pm 2, \dots$

then $\cos(2\pi f_0 N) = 1$, and $\sin(2\pi f_0 N) = 0$

giving us $x(n) = x(n+N)$.

Thus we must have $f_0 = \frac{k}{N}$ where

k and N are integers. That is f_0 must be a rational number.

* If $f_0 = \frac{k}{N}$, and k and N are integers, and $\frac{k}{N}$ can not be simplified any more the N , i.e. the denominator of $\frac{k}{N}$ is the fundamental period of $x(n)$.

Ex 2

Consider the following continuous-time signals.

(a) $x(t) = e^{-2t} u(t)$, (b) $x(t) = e^{-|t|}$, $-\infty < t < \infty$

(c) $x(t) = B \sin\left(200\pi t - \frac{\pi}{3}\right)$, $-\infty < t < \infty$

(d) $x(t) = u(t)$, (e) $x(t) = \delta(t)$,

(f) $x(t) = 5 + 3 \cos(300\pi t)$, $-\infty < t < \infty$

(g) $x(t) = r(t) u(t) + \sin(400\pi t)$, $-\infty < t < \infty$

(h) $x(t) = -2 \cos\left(2\pi\sqrt{300}t + \frac{\pi}{4}\right)$, $-\infty < t < \infty$

(i) $x(t) = \sqrt{2} + \cos(100\pi t) - e^{j(200\pi t - \frac{\pi}{6})}$

Now

(I) Find Discrete-Time or Sampled version of each signal $x(n)$ using sampling frequency of $F_s = 600$ Hz.

(II) Specify if $x(t)$ is Energy, Power, or periodic signal.

(III) Find fundamental period N for each periodic signal $x(n) = x(n+N)$, if $x(n)$ is periodic.

Solpart (I) let $t = nT_s = \frac{n}{F_s}$, T_s is sampling interval (or period)

$$\begin{aligned} \underline{(a)} \quad x(n) &= e^{-2nT_s} u(nT_s) \\ &= e^{-2\frac{n}{F_s}} u\left(\frac{n}{F_s}\right) \quad T_s \text{ is dropped for} \\ &= e^{-2n} u(n) \quad \text{simplicity of} \\ &\quad \text{notation.} \end{aligned}$$

$$\begin{aligned} \underline{(b)} \quad x(n) &= x(nT_s) = e^{-\frac{1}{T_s}|n|}, \quad -\infty < n < \infty \\ &= e^{-|n|} = e^{-\frac{|n|}{F_s}} = e^{-\frac{|n|}{600}}, \quad -\infty < n < \infty \end{aligned}$$

$$\begin{aligned} \underline{(c)} \quad x(n) &= B \sin\left(200\pi nT_s - \frac{\pi}{3}\right), \quad -\infty < n < \infty \\ &= B \sin\left(\frac{200\pi}{F_s} n - \frac{\pi}{3}\right) \\ &= B \sin\left(\frac{\pi}{3} n - \frac{\pi}{3}\right) = B \sin\left(\frac{\pi}{3}(n-1)\right) \end{aligned}$$

$$\underline{(d)} \quad x(n) = x(nT_s) = u(nT_s) = u(n)$$

$$\underline{(e)} \quad x(n) = x(nT_s) = \delta(nT_s) = \delta(n)$$

$$\begin{aligned} \underline{(f)} \quad x(n) &= x(nT_s) = 5 + 3 \cos(300\pi nT_s), \quad -\infty < n < \infty \\ &= 5 + 3 \cos\left(\frac{300}{F_s} \pi n\right) = 5 + 3 \cos\left(\frac{\pi}{2} n\right) \end{aligned}$$

$$\begin{aligned} \underline{(g)} \quad x(n) &= x(nT_s) = r(nT_s) u(nT_s) + \sin(400\pi nT_s) \\ &= r(n) u(n) + \sin\left(\frac{400\pi}{F_s} n\right) \\ &= r(n) u(n) + \sin\left(\frac{2\pi}{3} n\right) \quad -\infty < n < \infty \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad x(n) &= x(nT_s) = -2 \cos\left(2\pi\sqrt{3}T_s n + \frac{\pi}{4}\right) \\
 &= -2 \cos\left(\frac{2\pi\sqrt{300}}{600}n + \frac{\pi}{4}\right), \quad -\infty < n < \infty \\
 &\quad \text{j} \left(200\pi T_s n - \frac{\pi}{6}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad x(n) &= x(nT_s) = \sqrt{2} + \cos(100\pi T_s n) - e^{\text{j}(\frac{\pi}{3}n - \frac{\pi}{6})} \\
 &= \sqrt{2} + \cos\left(\frac{\pi}{6}n\right) - e^{\text{j}(\frac{\pi}{3}n - \frac{\pi}{6})} \\
 &\quad -\infty < n < \infty
 \end{aligned}$$

Part (II)

$$\begin{aligned}
 \text{(a)} \quad x(n) &= e^{-2n} u(n), \quad E_x = \sum_{n=-\infty}^{\infty} (\bar{e}^{-4})^n u(n) = \sum_{n=0}^{\infty} (\bar{e}^{-4})^n \\
 E_x &= \frac{1}{1 - \bar{e}^{-4}} < \infty, \quad \text{"Energy Signal"}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad x(n) &= \bar{e}^{-|n|}, \quad E_x = \sum_{n=-\infty}^{\infty} \bar{e}^{-2|n|} = \\
 E_x &= 1 + 2 \sum_{n=1}^{\infty} \bar{e}^{-2n} = 1 + 2 \frac{\bar{e}^{-2} - 0}{1 - \bar{e}^{-2}} = 1 + \frac{2\bar{e}^{-2}}{1 - \bar{e}^{-2}} \\
 &\quad \text{"Energy Signal"}
 \end{aligned}$$

(c) $x(n) = B \sin\left(\frac{\pi}{3}n - \frac{\pi}{3}\right)$ has infinite energy.
 To see if $x(n)$ is periodic, and to find N
 if it is periodic, we recall that

$$x(n) = B \sin\left(2\pi f n - \frac{\pi}{3}\right) = B \sin\left(\frac{\pi}{3}n - \frac{\pi}{3}\right)$$

gives $2\pi f = \frac{\pi}{3} \Rightarrow f = \frac{1}{6}$ is rational number.

Thus $N = 6$.

Thus

$$P_{x \text{ periodic}} = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{6} \sum_{n=0}^5 B^2 \sin^2\left(\frac{\pi}{3}n - \frac{\pi}{3}\right)$$

$$P_x = \frac{B^2}{6} \sum_{n=0}^5 \frac{1}{2j} \left(e^{j\left(\frac{\pi}{3}n - \frac{\pi}{3}\right)} - e^{-j\left(\frac{\pi}{3}n - \frac{\pi}{3}\right)} \right)$$

$$P_x = \frac{B^2}{6} \sum_{n=0}^5 \frac{1}{4} \left\{ 1 - e^{j\left(\frac{2\pi}{3}n - \frac{2\pi}{3}\right)} - e^{-j\left(\frac{2\pi}{3}n - \frac{2\pi}{3}\right)} + 1 \right\}$$

$$P_x = \frac{B^2}{24} \left\{ \sum_{n=0}^5 2 - e^{-j\frac{2\pi}{3}} \sum_{n=0}^5 \left(e^{j\frac{2\pi}{3}} \right)^n - e^{+j\frac{2\pi}{3}} \sum_{n=0}^5 \left(e^{-j\frac{2\pi}{3}} \right)^n \right\}$$

$$P_x = \frac{B^2}{24} \left\{ 12 - e^{-j\frac{2\pi}{3}} \frac{1 - e^{j\frac{2\pi}{3}6}}{1 - e^{j\frac{2\pi}{3}}} - e^{+j\frac{2\pi}{3}} \frac{1 - e^{-j\frac{2\pi}{3}6}}{1 - e^{-j\frac{2\pi}{3}}} \right\}$$

$$P_x = \frac{B^2}{24} \{ 12 \} = \frac{B^2}{2}$$

(d) $x(n) = u(n)$, This has infinite energy and is not periodic. To see if it is power signal:

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P_x = \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \right) \sum_{n=0}^N u(n) =$$

$$= \lim_{N \rightarrow \infty} \left(\frac{1}{2N+1} \right) (N+1) = \lim_{N \rightarrow \infty} \frac{N+1}{2N+1}$$

$$= \lim_{N \rightarrow \infty} \frac{N}{2N} = \frac{1}{2} \quad \text{"power signal"}$$

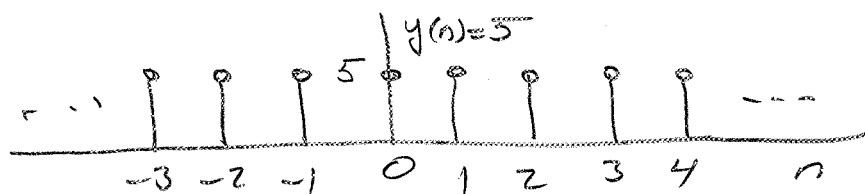
(e) $x(n) = \delta(n)$ is not periodic.

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |\delta(n)|^2 = 1$$

"Energy Signal"

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{elsewhere} \end{cases}$$

(f) $x(n) = 5 + 3 \cos\left(\frac{\pi}{2}n\right) \rightarrow$ Energy is infinite



Thus the first term on right is $y(n) = 5$. This is periodic with $N_y = 1$ as fundamental period.

The second term on the right is

$$v(n) = 3 \cos\left(\frac{\pi}{2}n\right). \text{ Check if } v(n) = v(n+N)$$

$$2\pi f = \frac{\pi}{2} \Rightarrow f = \frac{1}{4} \Rightarrow N_c = 4 \text{ is fundamental period of } v(n).$$

Now to find N for $x(n) = x(n+N)$:

$$N = k N_y = m N_c$$

$$\frac{N_c}{N_y} = \frac{m}{k} = \frac{4}{1} \Rightarrow \boxed{N = 4}$$

rational number

Thus $x(n) = x(n+4)$ and is periodic.

$$\begin{aligned}
 P_x &= \frac{1}{4} \sum_{n=0}^3 |x(n)|^2 = \frac{1}{4} \sum_{n=0}^3 \left(5 + 3 \cos\left(\frac{\pi}{2}n\right) \right)^2 \\
 &= \frac{1}{4} \sum_{n=0}^3 \left[25 + 30 \cos\left(\frac{\pi}{2}n\right) + 9 \cos^2\left(\frac{\pi}{2}n\right) \right] \\
 &= \frac{1}{4} \left\{ 100 + 0 + 9 \sum_{n=0}^3 \left[\frac{1}{2} + \frac{1}{2} \cos(\pi n) \right] \right\} \\
 P_x &= \frac{1}{4} \left\{ 100 + 18 + 0 \right\} = 25 + \frac{9}{2}.
 \end{aligned}$$

(9) $x(n) = r(n) u(n) + \sin\left(\frac{2\pi}{3}n\right)$, $-\infty < n < \infty$

$$\frac{2\pi}{3} = 2\pi f_s \Rightarrow f_s = \frac{1}{3} = \frac{1}{N_s} \Rightarrow N_s = 3$$

fundamental
period of sine
function.

But $r(n)u(n)$ is not periodic. Thus $x(n)$ is not periodic.

$y(n) = r(n)u(n)$ is Neither Energy signal nor power signal.

Thus $x(n)$ is neither energy nor power signal.

(h) $x(n) = -2 \cos\left(\frac{2\pi\sqrt{300}}{600}n + \frac{\pi}{4}\right), -\infty < n < \infty$

To check for periodicity:

$$\frac{2\pi\sqrt{300}}{600} = 2\pi f \Rightarrow f = \frac{\sqrt{300}}{600} \quad \text{This is}$$

not a rational number since

$$f = \frac{10\sqrt{3}}{600} = \frac{\sqrt{3}}{60}. \quad \text{Thus } x(n) \text{ is NOT periodic.}$$

Also the energy of this signal is infinite, thus it is not energy signal.

This signal is a power signal and has

Discrete Time Fourier Transform (DTFT) as we shall see later on.

(i) $x(n) = \sqrt{2} + \cos\left(\frac{\pi}{6}n\right) - e^{j\left(\frac{\pi}{3}n - \frac{\pi}{6}\right)}, -\infty < n < \infty$

To check for periodicity:

$y_1(n) = \sqrt{2}$ is periodic with period 1 = $N_{\text{const.}}$

$$y_2(n) = \cos\left(\frac{\pi}{6}n\right) \Rightarrow \frac{\pi}{6} = 2\pi f_c \Rightarrow f_c = \frac{1}{12}$$

$$y_3(n) = e^{j\left(\frac{\pi}{3}n - \frac{\pi}{6}\right)} \Rightarrow \frac{\pi}{3} = 2\pi f_c \Rightarrow f_c = \frac{1}{6}$$

$N_c = 12$
 $N_c = 6.$

Thus $x(n) = x(n+N)$ where $N=12 = N_c = 2N_e$
 $= 12 N_{\text{Const.}}$

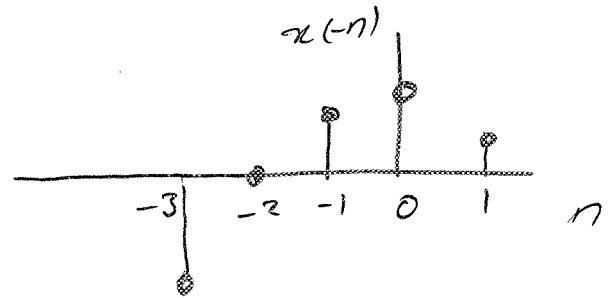
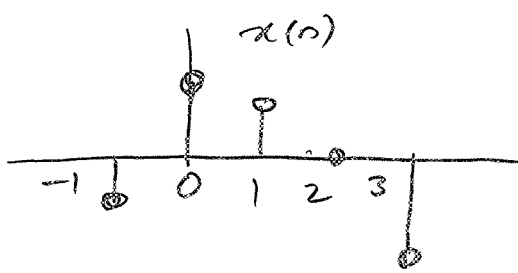
$$P_x = 2 + \frac{1}{2} - 1 = 1 + \frac{1}{2} = \frac{3}{2}.$$

Note $y_1(n)$, $y_2(n)$, and $y_3(n)$ are orthogonal to each other over period $N=12$. Thus power of $x(n)$ over one period ($N=12$) is sum of the power of each signal.



* Folding in time domain

$$x(nT_s) = x(n) \longrightarrow v(n) = x(-n) = v(nT_s) = x(-nT_s)$$



* Shifting in time

$$x(n) \longrightarrow v(n) = x(n-k) \text{ for } k > 0$$

shift to right
or delayed by k
sampling interval.

Note

$$x(n) = x(nT_s)$$

$$v(n) = x(nT_s \mp kT_s) \\ = x[(n \mp k)T_s]$$

$$g(n) = x(n+k)$$

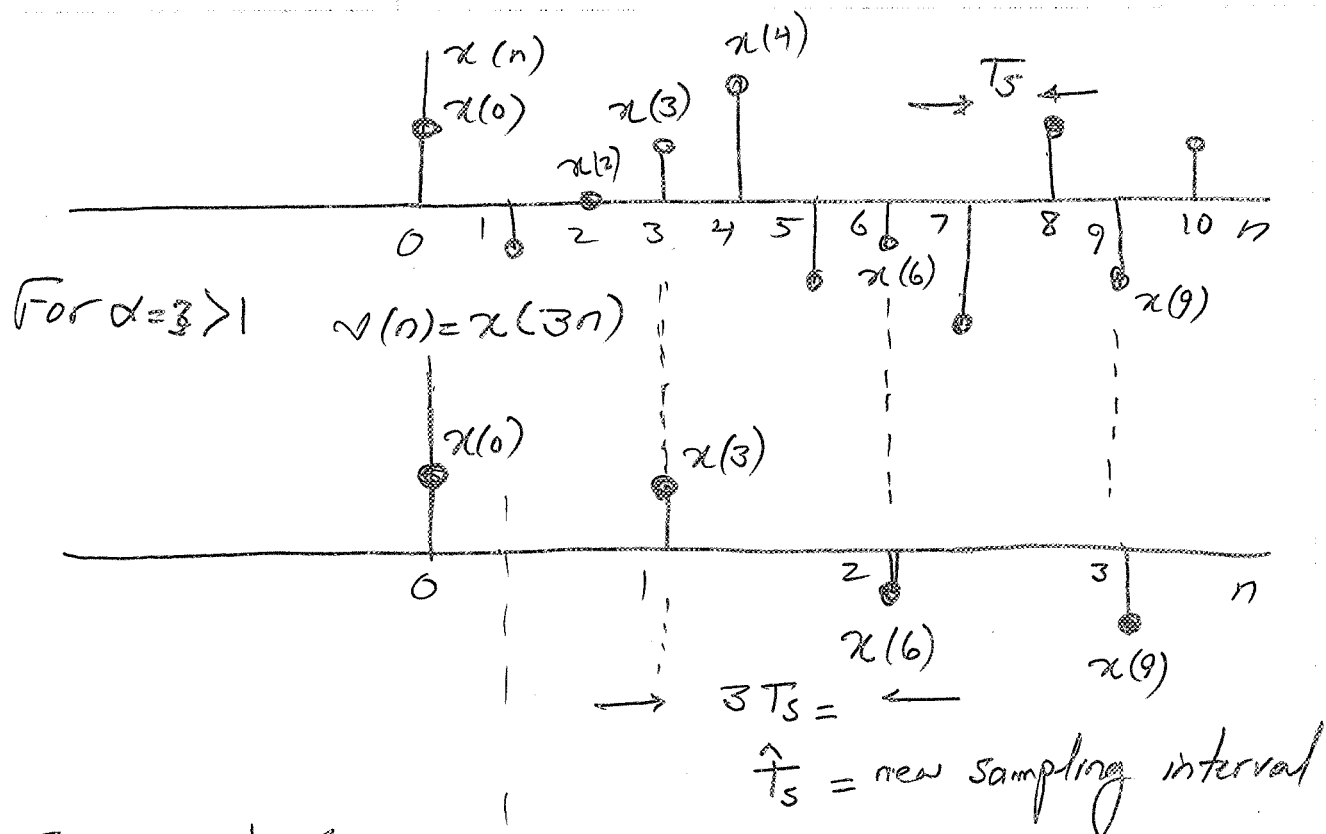
shift to left or
advanced by k
sampling interval.

* Time scaling

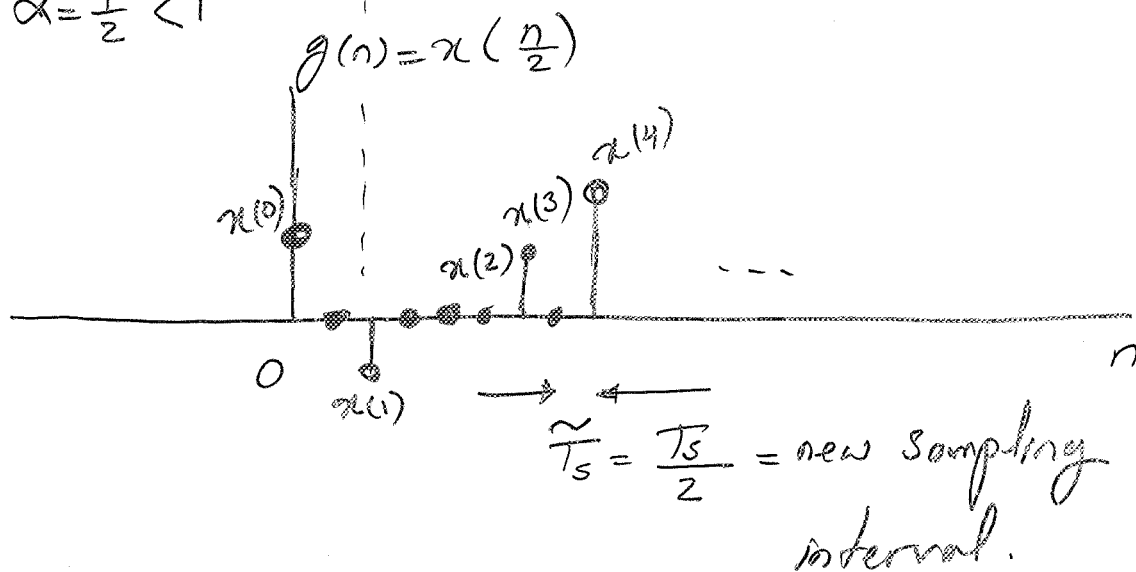
$$x(n) = x(nT_s)$$

$$v(n) = v(nT_s) = x(\alpha n) = x(n \alpha T_s)$$

That is sampling interval for $v(n)$ is
 α times the sampling interval of $x(n)$.

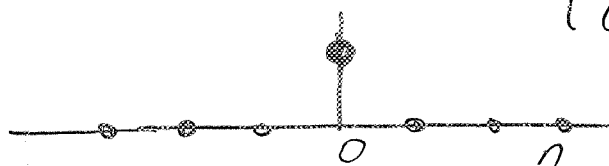


For $\alpha = \frac{1}{2} < 1$



Elementary Discrete-time Signals

$$(1) \quad \delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{elsewhere, or } n \neq 0 \end{cases}$$

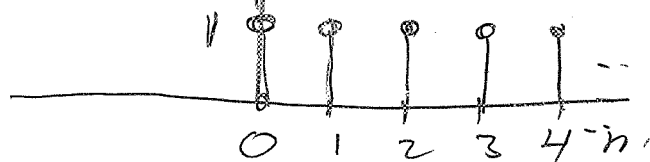


$\delta(n)$: Unit Sample Sequence

and
$$\delta(n-k) = \begin{cases} 1 & \text{for } n=k \\ 0 & \text{for } n \neq k \end{cases}$$

$$\delta(n+k) = \begin{cases} 1 & \text{for } n=-k \\ 0 & \text{for } n \neq -k \end{cases}$$

$$(2) \quad u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

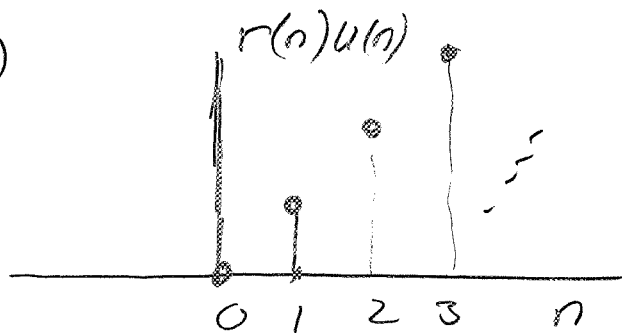


$u(n)$: Unit Step Sequence

Note

$$\delta(n) = u(n) - u(n-1)$$

$$(3) \quad r(n)u(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



$r(n)$: Ramp Sequence

(4) Exponential Sequence

$$x(n) = \rho^n e^{j\theta n} = (\rho e^{j\theta})^n$$

$$= \rho^n \cos(n\theta) + j\rho^n \sin(n\theta)$$

θn is $\angle x(n) \rightarrow$ phase sequence of $x(n)$

ρ^n is amplitude sequence of $x(n)$

Discrete-Time Systems



$$x(n) \xrightarrow{T} y(n)$$

Note

All classification and Definitions used for Continuous-time systems apply to Discrete-time Systems too.

- (1) T is Linear if superposition holds otherwise is non-linear:

$$\text{If } x_k(n) \xrightarrow{T} y_k(n) \text{ for } k=1,2,3,\dots,L$$

then

$$v(n) = \sum_{k=1}^L a_k x_k(n) \xrightarrow{T} y(n) = \sum_{k=1}^L a_k y_k(n)$$

"superposition property"

- (2) T is Time-Invariant if

$$\text{for any } x(n) \xrightarrow{T} y(n)$$

$$\text{then } v(n) = x(n \mp k) \xrightarrow{T} y(n) = y(n \mp k)$$

for any integer k .

(3) $\mathcal{T}$ is Bounded Input, Bounded Output (BIBO) stable iff

for any bounded input (i.e. $|x(n)| \leq M_x(\infty)$) the output is bounded (i.e. $|y(n)| \leq N_y(\infty)$).

(4) $\mathcal{T}$ is causal if output does not depend on the future values of input.

(5) $\mathcal{T}$ is at rest, or is Relaxed, or has zero initial conditions if there is no output before applying the input $x(n)$.

(6) $\mathcal{T}$ is static, or Memoryless, if output depends only on present values of input.

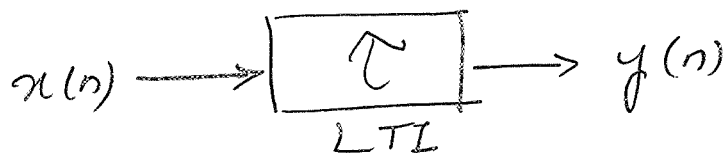
Examples:

(a) $y(n) = x(n^3)$ static, Non-causal, Linear,
time-varying, BIBO stable

(b) $y(n) = 5x(n) + 2x(n-3)$
Dynamic (with memory),
Causal, time-Invariant,
Linear, BIBO stable

(c) $y(n) = x^3(n)$ Non-Linear, Causal,
static (i.e. Memoryless),
time-Invariant, BIBO
stable

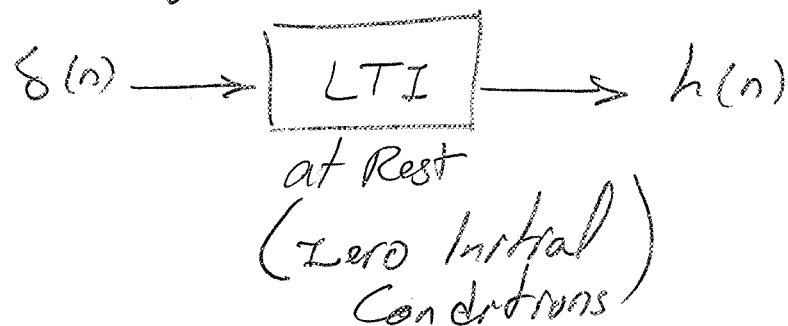
Impulse Response of LTI system and Convolution Sum



(1) τ is at rest

(2) $x(n) = \delta(n)$

Then the output is Impulse Response of the system τ , denoted by $h(n)$



Convolution Sum :

Note $x(n) \delta(n) = x(0) \delta(n)$

$$x(n) \delta(n-k) = x(k) \delta(n-k)$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k)$$

$$\delta(n) \xrightarrow{\mathcal{T}} h(n)$$

$$\delta(n-k) \xrightarrow{\mathcal{T}} h(n-k)$$

Time-Invariant
property

$$x(k) \delta(n-k) \xrightarrow{\mathcal{T}} x(k) h(n-k)$$

multiplication
property of
Superposition

for Linear
system

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k) \xrightarrow{\mathcal{T}} y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

Addition
property of
Superposition
for Linear
system

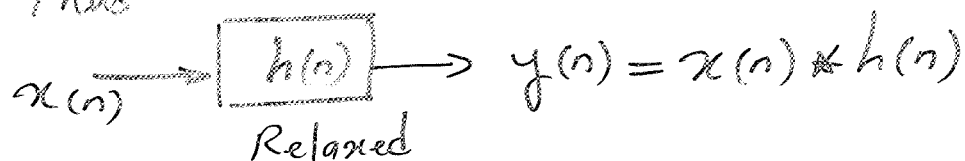
Thus

$$x(n) \xrightarrow[\text{LTI}]{\mathcal{T}} y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

This is Convolution Sum
for Discrete-Time
System.

$h(n)$ is Impulse response
of the LTI system $\mathcal{T}$.

Thus



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

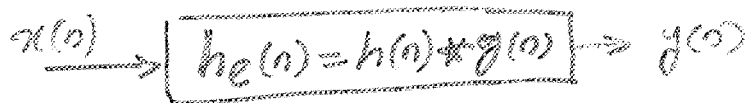
$$= \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

Properties of Convolution Sum

(1) Commutative

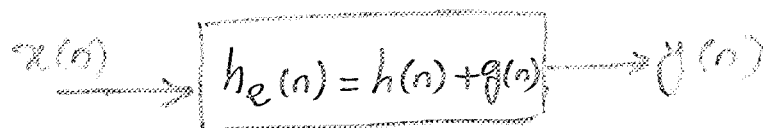
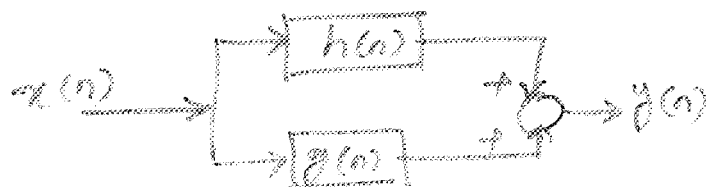
$$y(n) = x(n) * h(n) * g(n) = h(n) * x(n) * g(n)$$

In any order, for any number of signals



(2) Associative

$$y(n) = x(n) * h(n) + x(n) * g(n) = x(n) * [h(n) + g(n)]$$



(3) Linearity

$$y_k(n) = x_k(n) * h(n) \quad k=1, 2, 3, \dots, L$$

$$\begin{aligned} v(n) &= \sum_{k=1}^L a_k y_k(n) = h(n) * \sum_{k=1}^L a_k x_k(n) \\ &= h(n) * x(n) \end{aligned}$$

$$\text{where } x(n) = \sum_{k=1}^L a_k x_k(n)$$

(4) Shifting (Time-Invariant)

$$\text{If } y(n) = x(n) * h(n)$$

then

$$v(n) = y(n-\alpha-\beta) = x(n-\alpha) * h(n-\beta)$$

Ex 1

$$x(n) = a^{-n} u(n), \quad h(n) = b^{-n} u(n) \quad |a| > 1, \quad |b| > 1.$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{a^{-k}}_{k \geq 0} \underbrace{b^{-n+k}}_{n-k \geq 0} u(n-k)$$

$$y(n) = \sum_{k=0}^n a^{-k} b^{-n} b^k = b^{-n} \sum_{k=0}^n \left(\frac{b}{a}\right)^k$$

$$y(n) = b^{-n} \frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \left(\frac{b}{a}\right)} = b^{-n} \frac{a^{n+1} - b^{n+1}}{a^{n+1} - b a^n}$$

$$y(n) = \frac{a^{n+1} - b^{n+1}}{b^n a^{n+1} - b^{n+1} a^n}$$

Ex 2

$$x(n) = a^{-n+3} u(n-3), \quad h(n) = b^{-n+4} u(n-4)$$

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} a^{-k+3} u(k-3) b^{-n+4+k} u(n-k-4)$$

$$\text{Let } k-3 = \alpha, \quad k = \alpha+3$$

$$y(n) = \sum_{\alpha=-\infty}^{\infty} a^{-\alpha} u(\alpha) b^{-n+4+\alpha+3} u(n-\alpha-3-4)$$

$$y(n) = \sum_{\alpha=-\infty}^{\infty} a^{-\alpha} u(\alpha) b^{-n+\alpha+7} u(n-\alpha-7)$$

for $n \geq 0$

Let $n-7 = n$, then

$$y(n-7) = \sum_{\alpha=-\infty}^{\infty} \underbrace{a^{-\alpha}}_{\alpha \geq 0} \underbrace{b^{-n+\alpha}}_{n \geq \alpha} \underbrace{u(n-\alpha)}_{\alpha \geq 0}$$

for $n \geq 7$

$$y(n-7) = \sum_{\alpha=0}^n a^{-\alpha} b^{-n+\alpha} = b^{-n} \sum_{\alpha=0}^n \left(\frac{b}{a}\right)^{\alpha}$$

$$y(n-7) = b^{-n} \frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \left(\frac{b}{a}\right)} \quad \text{for } n \geq 7$$

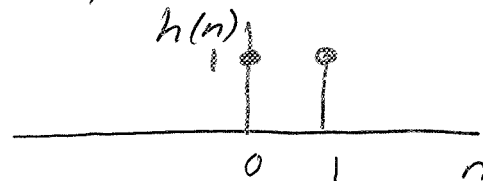
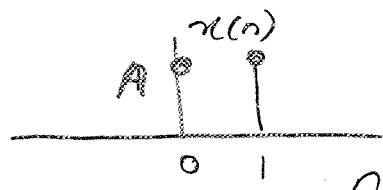
or

$$y(n) = y(n-7) = b^{-n} \frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \left(\frac{b}{a}\right)} u(n-7)$$

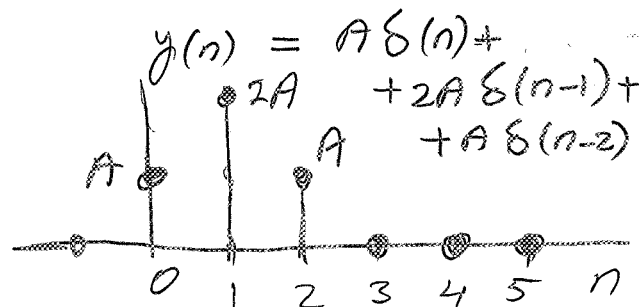
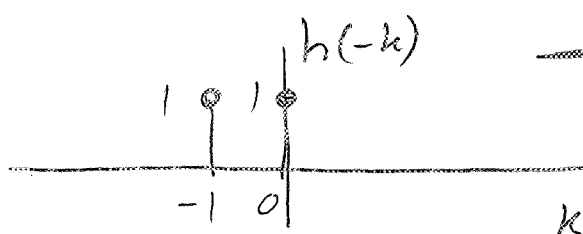
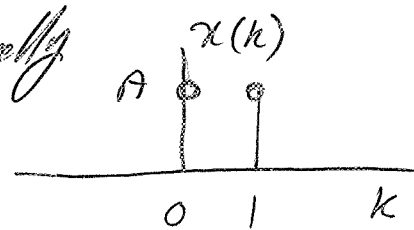
$$y(n) = 0 \quad \text{for } n < 7.$$

Ex 3

$$x(n) = A[u(n) - u(n-2)], \quad h(n) = u(n) - u(n-2)$$



Graphically



$$y(n) = A\delta(n) + 2A\delta(n-1) + A\delta(n-2)$$

Ex 4

$$x(n) = A u(n), \quad h(n) = B u(n) + \delta(n-2)$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} A u(k) [B u(n-k) + \delta(n-k-2)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} \left\{ AB u(k) u(n-k) + A u(k) \delta(n-k-2) \right\}$$

$y(n) = 0$ for $n < 0$; It is easy to see this graphically.

For $n \geq 0$:

$$y(n) = AB \sum_{k=0}^n 1 + A \sum_{k=-\infty}^{\infty} u(k) \delta(n-k-2)$$

$$y(n) = AB(n+1) + A u(n-2) \quad \text{for } n \geq 0.$$

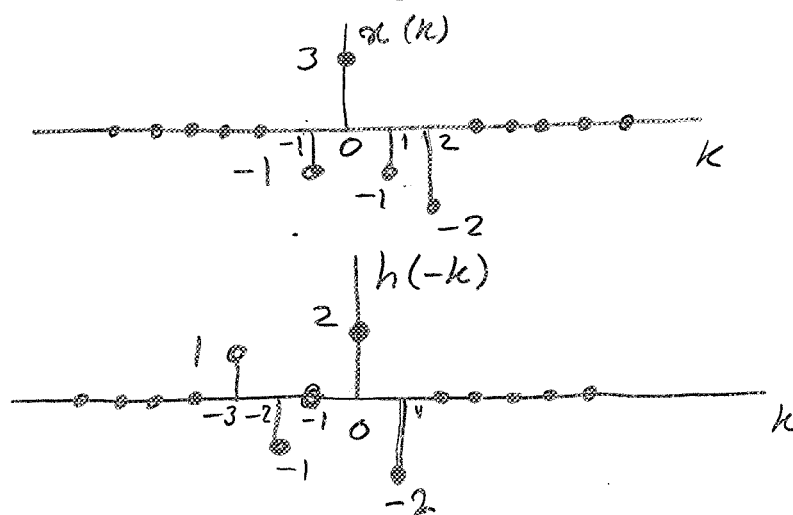
Ex 5 $x(n) = \{-1, \underset{\uparrow}{3}, -1, -2\}$, $h(n) = \{-2, \underset{\uparrow}{2}, 0, -1, 1\}$. Find

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \text{ for all } n \text{ (i.e. } -\infty < n < \infty)$$

Note If length of $x(n)$ is L and length of $h(n)$ is M , then $y(n) = x(n) * h(n)$ will be finite sequence of length $L+M-1$.

To find $y(n) = x(n) * h(n)$ we note that $L=4$, and $M=5$. Thus $y(n)$ is of length $L+M-1=8$.

Also notice that $h(n) = h(n) [u(n+1) - u(n-4)]$
and $x(n) = x(n) [u(n+1) - u(n-3)]$



$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-1}^2 x(k) h(n-k)$$

$y(n) = 0$ for $n < -2$ because

$$y(-3) = \sum_{k=-1}^2 x(k) h(-3-k)$$

$$y(-3) = \underbrace{x(-1)h(-2)}_0 + \underbrace{x(0)h(-3)}_0 + \underbrace{x(1)h(-4)}_0 + \dots$$

so is for $y(-4)$, $y(-5)$, ... $+ \underbrace{x(2)h(-5)}_0$

and for $y(n < -2)$.

Now

$$n = -2 \quad y(-2) = \sum_{k=-1}^2 x(k)h(-2-k)$$

$$= x(-1)h(-1) + x(0)h(-2) + x(1)h(-3)$$

$$y(-2) = x(-1)h(-1) = (-1)(-2) = 2 \quad + x(2)h(-4)_0$$

$$n = -1 \quad y(-1) = \sum_{k=-1}^2 x(k)h(-1-k)$$

$$= x(-1)h(0) + x(0)h(-1) + x(1)h(-2) + x(2)h(-3)_0$$

$$y(-1) = (-1)(2) + (3)(-2) = -8$$

$$n = 0 \quad y(0) = \sum_{k=-1}^2 x(k)h(-k) = x(-1)h(1) + x(0)h(0)$$

$$+ x(1)h(-1) + x(2)h(-2)_0$$

$$y(0) = (-1)(0) + (3)(2) + (-1)(-2) = 8$$

$$n = 1 \quad y(1) = \sum_{k=-1}^2 x(k)h(1-k)$$

$$= x(-1)h(2) + x(0)h(1) + x(1)h(0) + x(2)h(-1)$$

$$= (-1)(-1) + (3)(0) + (-1)(2) + (-2)(-2)$$

$$y(1) = 1 + 0 - 2 + 4 = 3$$

$$\begin{aligned}
 n=2 \quad y(2) &= \sum_{k=-1}^2 x(k) h(2-k) \\
 &= x(-1)h(3) + x(0)h(2) + x(1)h(1) + x(2)h(0) \\
 &= (-1)(1) + (3)(-1) + (-1)(0) + (-2)(2)
 \end{aligned}$$

$$y(2) = -1 - 3 + 0 - 4 = -8$$

$$\begin{aligned}
 n=3 \quad y(3) &= \sum_{k=-1}^2 x(k) h(3-k) \\
 &= x(-1) \underbrace{h(4)}_0 + x(0)h(3) + x(1)h(2) + x(2)h(1) \\
 &= 0 + (3)(1) + (-1)(-1) + (-2)(0)
 \end{aligned}$$

$$y(3) = 0 + 3 + 1 + 0 = 4$$

$$\begin{aligned}
 n=4 \quad y(4) &= \sum_{k=-1}^2 x(k) h(4-k) \\
 &= x(-1) \underbrace{h(5)}_0 + x(0) \underbrace{h(4)}_0 + x(1)h(3) + x(2)h(2)
 \end{aligned}$$

$$y(4) = (-1)(1) + (-2)(-1) = -1 + 2 = 1$$

$$\begin{aligned}
 n=5 \quad y(5) &= \sum_{k=-1}^2 x(k) h(5-k) \\
 &= x(-1) \underbrace{h(6)}_0 + x(0) \underbrace{h(5)}_0 + x(1) \underbrace{h(4)}_0 + x(2)h(3)
 \end{aligned}$$

$$y(5) = (-2)(1) = -2$$

$$\begin{aligned}
 n=6 \quad y(6) &= \sum_{k=-1}^2 x(k) h(6-k) = x(-1) \underbrace{h(7)}_0 + x(0) \underbrace{h(6)}_0 \\
 &\quad + x(1) \underbrace{h(5)}_0 + x(2) \underbrace{h(4)}_0
 \end{aligned}$$

$$y(6) = 0$$

Thus $y(n) = 0$ for $n \geq 6$.